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Quantum Aspects of the Fundamental Dirac Membrane Model

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The classical relativistic Hamiltonian derived by Dirac for a charged membrane is written in a linearized form and it is pointed out that the membrane has spin 1/2 under the action of an external magnetic field. A spin-rotation coupling term is included into the linearized Hamiltonian and the corresponding wave equation for the membrane is written. It leads to quantized radial modes of oscillations and its first eigenvalues are derived numerically. Asymptotic solutions are also considered.

KEY WORDS: charged membrane; Dirac equation; numerical solution.

1. PRELIMINARY REVIEW

There has been recently a rapidly growing interest in relativistic membrane theories. Also, there are several extended models for particles, where each one attempts to describe an elementary particle based on an Hamiltonian formulation. We recall that the basic idea of an extended electron in the context of a relativistic membrane was formulated by Dirac (1962), with the purpose to explain the origin of the muon. Dirac has assumed a charged spherical membrane as a manifold imbedded in the Minkowski space time, where the coulumbic repulsion between elements of such a "closed oscillating bubble" is supposed to be balanced by a type of surface tension. By means of a semiclassical quantization, he has pointed out that, owing to radial oscillations, the first excited state has an energy lower than 25% of the muon rest mass. Since the above treatment is not trustworthy, such an approach lacks an accurate relativistic quantum formulation. Collins and Tucker (1976) studied later the classical mechanics of uncharged relativistic membranes, and afterwards Howe and Tucker (1977) considered a locally super symmetric and reparametrization-invariant action for a spinning membrane. Such a model is the membrane analog of the Neveu–Schwarz–Ramond formulation of a spinning string. However, there are several unclear points in this latter membrane framework.

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Before describing a quantum approach for a charged closed membrane, we must face an important conceptual problem. Since the membrane has an uniform distribution of charge on its surface, its Lagrangian may be interpreted as providing repulsive forces at any instant between the elements of the membrane properly, which are balanced by a surface tension. However, if the membrane rotates about an arbitrary axis, owing to the influence of an external magnetic field, then points of its instantaneous axis of rotation experience no centrifugal reaction (Collins and Tucker, 1976). Therefore, we expect that the membrane should oscillate about the rotation axis and then under quantization such oscillations might have a ground state, defined at the equilibrium configuration of the membrane. An asymptotic distribution of mass spectrum for sphere-like membranes immersed in a Minkowski space-time has been studied by Ho (1995), and the spectrum of the Dirac membrane model has been derived by means of a semiclassical quantization, in agreement with the previous results obtained by Dirac (1962).

A covariant formulation for a moving charged membrane in an arbitrary dimension and coupled to an electromagnetic field has been studied by Barut and Pavsic (1993), where in a particular case a new formulation of Dirac's model of the electron as a charged spherical shell with internal oscillations and finite self-energy was obtained. Afterwards the same authors (Barut and Pavsic 1994) pointed out that the Dirac membrane approach provides a stable electron model with finite self-energy, whose stability is owing to the surface tension of the membrane. The stability of isotropic spherical rotating membranes has been studied recently by Axenides *et al*. (2001), with a generalization to anisotropic ellipsoidal membranes, where it was pointed out that ellipsoidal rotating membranes generally decay into finger-like configurations.

This paper is devoted to the study of a charged membrane under the action of an external magnetic field, within a relativistic quantum framework. First, we linearize the Hamiltonian of the Dirac free membrane and then study its spin when a membrane is immersed in a magnetic field. Afterwards we include into the linearized Hamiltonian operator an interaction term, which contains a spin-rotation coupling. Finally, some eigenvalues of the corresponding wave equation are derived numerically.

2. THE HAMILTONIAN OPERATOR

Let us start from the classical relativistic Hamiltonian derived by Dirac (1962) for a charged spherical membrane in the absence of external fields

$$H = \sqrt{p_r^2 + \mu^2(r)} + V(r), \tag{1}$$

where $\boldsymbol{p}_r$ is the radial momentum, $\mu(r) = \gamma r^2/4a^3$, $V(r) = \gamma/2r$ is a coulumbic self-interaction potential and a is the electron classical radius. Here we adopt a system of units where $\hbar = c = 1$ and $\gamma = e^2 = 1/137$ is the fine structure constant.

The classical relativistic Hamiltonian (1) for a free membrane has as kinetic term the positive square root of $p_r^2 + \mu^2$, and therefore if p_r is replaced by $-i\partial/\partial r$ and the above kinetic term is substituted into the Schrodinger equation, the resulting wave equation is unsymmetrical with respect to space and time derivatives, and hence it is not relativistic. For that reason we have to modify the Hamiltonian (1) in such a way as to make it linear as an operator in space derivatives. We thus extend the Dirac method of linearization, searching for the linearized Hamiltonian in the standard from $H = T + V(r)$, where $T = +\sqrt{p_r^2 + \mu^2}$. In this procedure we take into account the Hermitian matrices

$$\alpha = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{I} \end{pmatrix},$$

where σ are the Pauli matrices and $\boldsymbol{I}$ is the identity matrix. We choose to write T in the matrix form

$$\boldsymbol{T} = \alpha \cdot (p_r + iq) + \beta\mu, \tag{2}$$

where the momentum p_r is modified according to the gauge transformation $p_r \to p_r + iq$, and q is a real radial vector, whose component is an arbitrary function of r. With the help of Eq. (2) we find

$$\begin{aligned} \boldsymbol{T}^\dagger \boldsymbol{T} = \left(p_r^2 + \mu^2 + q^2\right) \boldsymbol{I} + i[(\alpha \cdot p_r)(\alpha \cdot q) - (\alpha \cdot q)(\alpha \cdot p_r)] \\ + (\alpha \cdot p_r)\beta\mu + \beta\mu(\alpha \cdot p_r) + i[\beta\mu(\alpha \cdot q) - (\alpha \cdot q)\beta\mu]. \end{aligned} \tag{3}$$

Assuming that q commutes with $\mu(r)$ and since the linearization we search for requires that $T^2 = I(p_r^2 + \mu^2)$, we are led to a system of equations in spherical coordinates

$$\nabla \cdot q + q^2 = 0, \qquad \sigma \cdot (\nabla_r \mu) + 2\mu(\sigma \cdot q) = 0, \tag{4}$$

whose solution is $q = -e_r/r$. Henceforth, the Hamiltonian (1) is transformed into an operator in the linearized form

$$\boldsymbol{H} = \alpha \cdot \left[p_r - i\frac{e_r}{r}\right] + \beta\mu + \boldsymbol{I}V, \tag{5}$$

where the term $\alpha \cdot [p_r - ie_r/r]$ is an Hermitian operator (Schiff, 1968), and therefore the Hamiltonian (5) is also Hermitian.

We recall that as stated in Dirac's original paper (Dirac, 1962), the ground state of the charged spherical membrane is obtained from the classical Hamiltonian (1), at the equilibrium configuration where $p_r = 0$ and $r = a$. This yields the classical energy $\epsilon_0 = 3\gamma/4a$, which is the electron rest energy. The same result is also obtained from the Hamiltonian (5), since $p_r = 0$ leads to $q = 0$, according to Eq. (4).

3. SPIN OF THE MEMBRANE

With the purpose to study a relativistic quantum framework for the Dirac membrane, let us now assume that it is spinning about an arbitrary axis with angular momentum Ω, owing to the influence of an external magnetic field. However, under radial oscillations and rotation, the membrane poles on the instantaneous axis of rotation experience no centrifugal reaction. As a consequence, the original spherical shell is deformed into a triaxial ellipsoidal shell with sharp-pointed poles. Once the Hamiltonian (5) is linear in the radial derivative, we can insert the term $\alpha \cdot (p_\theta e_\theta + p_\phi e_\phi)$ into its kinetic part, and then we can write

$$\boldsymbol{H} = \alpha \cdot P + \beta\mu + IV, \tag{6}$$

where $P = p - ie_r/r$ and $p = p_r e_r + p_\theta e_\theta + p_\phi e_\phi$ in a spherical left-handed basis (e_r, e_ϕ, e_θ).

For simplicity, let us assume that the membrane is spinning around the Z-axis and that it is instantaneously without translational motion. We point out below that Ω is not a constant of the motion. For that, we calculate its time rate of change in the Heisenberg picture in Cartesian coordinates. First, we notice that the commutation rules $[\Omega_j, P_k] = [\Omega_j, p_k] = i\epsilon_{jk\ell\, p\ell}$ and $[\Omega_j, \mu(r)] = [\Omega_j, V(r)] = 0$, $(j, k, \ell = 1, 2, 3)$ hold true, because $\mu(r)$ and $V(r)$ are spherically symmetric functions. Since $\Omega_x = \Omega_y = 0$, hence for the Ω_z component we find

$$[\Omega_z, \boldsymbol{H}] = [\Omega_z, p_x]\alpha_x + [\Omega_z, p_y]\alpha_y = i(\alpha_x p_y - \alpha_y p_x). \tag{7}$$

Now, let us take into account the Hermitian matrix

$$\Sigma = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix},$$

which satisfies the commutation rules $[\Sigma_j, \alpha_k] = 2i\varepsilon_{jk\ell}\,\alpha_\ell$, $[\Sigma_j, \beta] = [\Sigma_j, \mu_(r)] = [\Sigma_j, V(r)] = 0$. If we evaluate the time rate of change of Σ_z we obtain

$$[\Sigma_z, \boldsymbol{H}] = [\Sigma_z, \alpha_x]p_x + [\Sigma_z, \alpha_y]p_y = 2i(\alpha_y p_x = \alpha_x p_y). \tag{8}$$

Therefore, we conclude from Eqs. (7) and (8) that we can define the quantity

$$J_z = \boldsymbol{I}\Omega_z + \frac{1}{2}\Sigma_z = \Omega_z + S_z, \tag{9}$$

in such a way that $dJ_z/dt = \mathbf{0}$, because it commutes with the Hamiltonian operator of the membrane and so it is a constant of the motion. Moreover, the operator S_z in Eq. (9) is

$$S_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{10}$$

and since the eigenvalues of an operator represented by a diagonal matrix are the same values of its diagonal elements, then the eigenvalues of S_z are $\pm 1/2$. From the above results, we conclude that it is apparent that the same conclusions hold for the components $\boldsymbol{J}_x$ and $\boldsymbol{J}_y$. Hence, we can define the total angular momentum operator of the membrane by $\boldsymbol{J} = \Omega + (1/2)\Sigma = \Omega + \boldsymbol{S}$, where $\boldsymbol{S}$ is the spin angular momentum operator. The above result is in agreement with the fact that spin 1/2 requires a nontrivial homotopy under 2π rotation, which a spherical membrane lacks (Hu, 1959). Notice that the trivial homotopy is broken owing to the existence of sharp-pointed poles of the ellipsoidal shell.

4. THE WAVE EQUATION

Once the charged membrane is spinning under the influence of an external magnetic field, then the interaction between its spin and the external field should be included into its Hamiltonian operator. Such an interaction leads to a spin-rotation coupling, which can be represented by an Hamiltonian operator $H_{\text{int}} \approx 2(\Omega \cdot S) = J^2 - \Omega^2 - S^2$, where $J^2 = j(j+1)$ with $j = 1/2, 3/2, \ldots$, $\Omega^2 = \ell(\ell+1)$, $\ell = 0, 1, 2, \ldots$ and $S^2 = 3/4$. Lets us choose the case $j = 1/2$ and let us derive the corresponding wave equation. With these assumptions and making analogy to the relativistic case of an electron in an external magnetic field (Davydov, 1965), and also the case of a closed charged shell spinning in an uniform magnetic field with a given angular frequency, we choose to write with the help of Eq. (6)

$$\boldsymbol{H} = \boldsymbol{\alpha} \cdot \boldsymbol{P} + \beta\mu + \boldsymbol{I}V\frac{2\pi\gamma}{a}\beta\boldsymbol{\Omega}^2, \tag{11}$$

where the latter term in the above equation is Hermitian.

Let us now consider the wave equation of the membrane as an eigenvalue equation for the wave function $\boldsymbol{\Psi}(r, \theta, t) = \boldsymbol{\Psi}(r, \theta, \phi)\exp(-iEt)$ in the matrix form

$$\boldsymbol{H}\boldsymbol{\Psi}(r, \theta, \phi, t) = E\Psi(r, \theta, \phi, t), \tag{12}$$

where E are the eigenvalues of $\boldsymbol{H}$ and the eigenstates of the membrane result from solving the above differential equation assuming the Hamiltonian operator (11). In a 2×2 representation

$$\boldsymbol{\Psi}(r, \theta, \phi) = \begin{pmatrix} \varphi(r, \theta, \phi) \\ \chi(r, \theta, \phi) \end{pmatrix},$$

we find from Eqs. (11) and (12) the pair of first order equations

$$(\boldsymbol{\sigma} \cdot P)\chi + \left[\mu + V - E - \frac{2\pi\gamma}{a}\Omega^2\right]\varphi = 0, \tag{13}$$

$$(\boldsymbol{\sigma}\cdot P)\varphi+\left[V-E-\mu+\frac{2\pi\gamma}{a}\Omega^2\right]\chi=0, \tag{14}$$

and therefore, states with well-defined values of both the radial coordinate and momentum of the membrane are described by the system of Eqs. (13) and (14).

Let us multiply on the right the Hermitian conjugate of Eq. (13) by $[V-E-\mu+(2\pi\gamma/a)\Omega^2]$, and let us consider the Hermitian conjugate of Eq. (14). Assuming the operator identities $[e_r/r]\cdot p+p\cdot[e_r/r]=-i\nabla(e_r/r)+2(e_r/r)\cdot p$ and $\nabla\times(e_r/r)=0$, we find a second order iterated equation

$$-\left[p^2+\frac{2}{r}\frac{\partial}{\partial r}\right]\varphi+\left[E^2-2VE+V^2-\mu^2+\frac{4\pi\gamma\mu}{a}\Omega^2-\frac{4\pi^2\gamma^2}{a^2}\Omega^4\right]\varphi=0. \tag{15}$$

Let us write now the spatial part of the wave function in the separable form $\varphi(r,\theta,\phi)=\boldsymbol{R}(r)F(\theta)G(\phi)$ and let us take into account that $-p^2=\nabla^2=\nabla_r^2-\Omega^2/r^2$, where $\nabla_r^2=(1/r^2)\,\partial/\partial r(r^2\partial/\partial r)$ is the radial part of the Laplace operator. Finally, let us consider the dimensionless quantities $X=r/a$ and $\varepsilon=aE$, and let us make the transformation $\boldsymbol{R}(r)=\boldsymbol{Y}(X)$. With the above assumptions we obtain from Eq. (15)

$$\frac{d^2\boldsymbol{Y}}{d\boldsymbol{X}^2}+\left[\varepsilon^2-\gamma\frac{\varepsilon}{X}+\frac{\gamma^2-4\ell(\ell+1)}{4X^2}-\frac{\gamma^2}{16}X^4+\pi\gamma^2\ell(\ell+1)X^2-4\pi^2\gamma^2\ell^2(\ell+1)^2\right]Y=0. \tag{16}$$

The above equation is the wave equation of an anharmonic oscillator with perturbative terms. The third term between brackets is a "centrifugal potential" that defines a "centrifugal barrier," which does not allow the membrane to collapse to a point and the fourth term is a quartic nonharmonic term; the fifth term between brackets is a harmonic term and the latter one is due to the spin-rotation coupling. Therefore, the insertion of the interaction Hamiltonian according to Eq. (11) gives rise to a harmonic term and also to a spin-rotation coupling term in the wave equation, in analogy as it happens in the so-called Dirac oscillator (Moshinsky and Szczepaniak, 1989). Otherwise, we recall that at the equilibrium radius $r=a$ (i.e., $X=1$), the radial momentum of the membrane vanishes, i.e. $\boldsymbol{Y}'(X=1)=0$ and therefore $\boldsymbol{Y}''(X=1)=0$. Moreover, if we assume $\ell=0$, we find from Eq. (16) the solution $E_0=3\gamma/4a$, which is the electron rest energy, in agreement to the classical result.

The eigenvalues ε of Eq. (16) are here derived by means of a numerical technique, where we assume the cases $\ell=0$ and $\ell=1$. We employ the Runge–Kutta method imposing the boundary condition $\boldsymbol{Y}'(0)$ as an arbitrary value, because the scaling procedure used in the derivation of Eq. (16) makes its numerical

solutions independent from the choice of the first derivative of $\boldsymbol{Y}(X)$ at the origin (Press *et al.*, 1992). We start from the boundary conditions $Y(0) = 0$ and $Y'(0) = N$, where N denotes an arbitrary value. Here our problem is reduced to find out $Y'(X) = Y(X) = 0$ for several values of $\mathcal{E}$. This procedure is similar to the usual one accounted for in the numerical solutions of the wave equation of a three-dimensional isotropic harmonic oscillator endowed with a centrifugal potential (Giordano 1997). Henceforth, we are tempted to shoot many values for $\mathcal{E}$ in Eq. (16) until we obtain numerically the desired asymptotic behaviors of both $\boldsymbol{Y}(X)$ and $\boldsymbol{Y}'(X)$. The Fortran program we have employed gives numerically the eigenfunctions $Y'n(X)$ in a normalized form in terms of the corresponding eigenvalues of Eq. (16) with reasonable accuracy. In Figs. 1(a)–(h) we show the plots of the first eigenfunctions $\boldsymbol{Y}_n(X)$ of the excited states in terms of the corresponding eigenvalues $\mathcal{E}_n^{(\ell)}$ of Eq. (16). If we now assume that the ground state energy of the charged membrane is the electron rest energy, we find $\mathcal{E}_0 = aE_0 = 3\gamma/4 = 0.00547$, and since the energy level of nth order is given by $\mathcal{E}_n^{(\ell)} = aE_n^{(\ell)}$, we obtain with $E_0 = 0.5110$ MeV,

$$E_n^{(\ell)} = 93.4186\mathcal{E}_n^{(\ell)} \text{ (MeV)}. \tag{17}$$

In Table I, it is shown the eigenvalues of Eq. (16) and the corresponding excited energies as multiple values of the ground state energy, for $\ell = 0$ and $\ell = 1$.

Let us now evaluate the behavior of Eq. (16) for very large values of $\mathcal{E}$. Since in this case the numerical solution is not trustworthy, owing to the great oscillatory behavior of the wave function, we take Eq. (16) in the asymptotic form

$$\frac{d^2\boldsymbol{Y}}{dX^2} + \mathcal{E}^2\boldsymbol{Y} = 0, \tag{18}$$

and we assume the boundary conditions $\boldsymbol{Y}(0) = 0$ and $\boldsymbol{Y}(X) = \boldsymbol{0}$ for large values of $\boldsymbol{X}$. We thus obtain

$$\boldsymbol{Y}(X) = \boldsymbol{I}A\sin(\mathcal{E}X), \tag{19}$$

where A is a normalization constant, and $\mathcal{E}_m^{(h)} = m\pi$, $(m = 1, 2, 3, \ldots)$ are the corresponding eigenvalues of high-order excited states. The first eight high-order excited states are shown in Table II.

Concluding this section let us consider the mass spectrum in the Dirac membrane model, derived by means of the Bohr–Sommerfeld quantization rule Ho (1995)

$$m = \frac{m_e}{3}\left(\frac{4}{e^2}\right)^{2/3}\left[\frac{n\pi}{\int_0^1\sqrt{1-u^4}\,du}\right]^{2/3}, \quad n = 1, 2, 3, \ldots \tag{20}$$

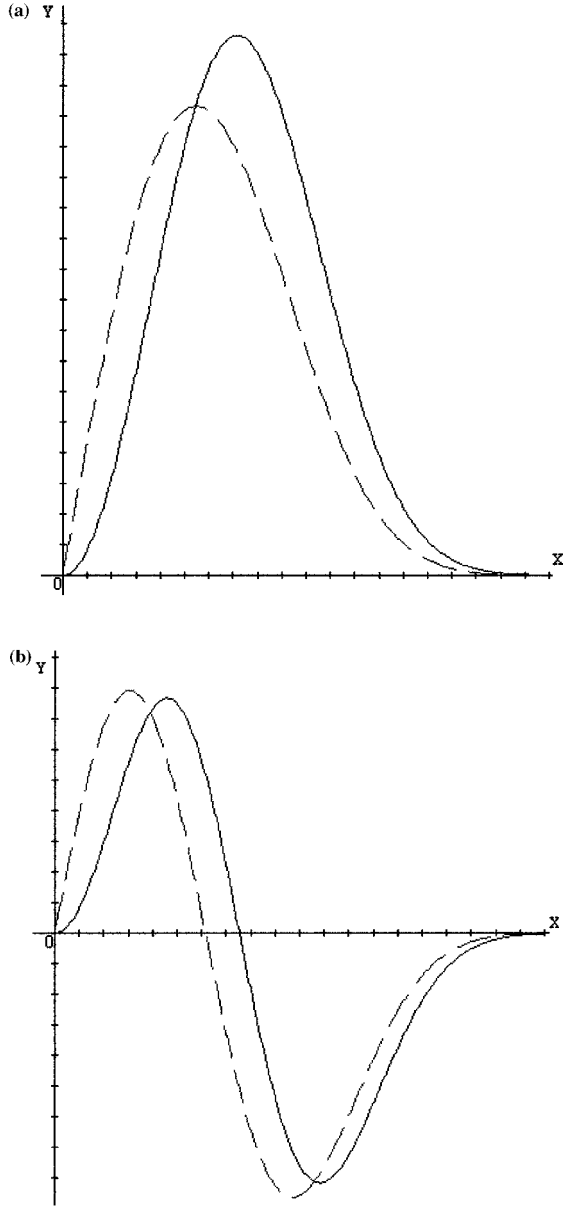

Fig. 1. (a–h) First eigenfunctions of the excited low-order energy levels of a charged membrane in an external magnetic field, in terms of the corresponding eigenvalues $\mathcal{E}_n^{(0)}$ (dashed line) and $\mathcal{E}_n^{(1)}$ (solid line).

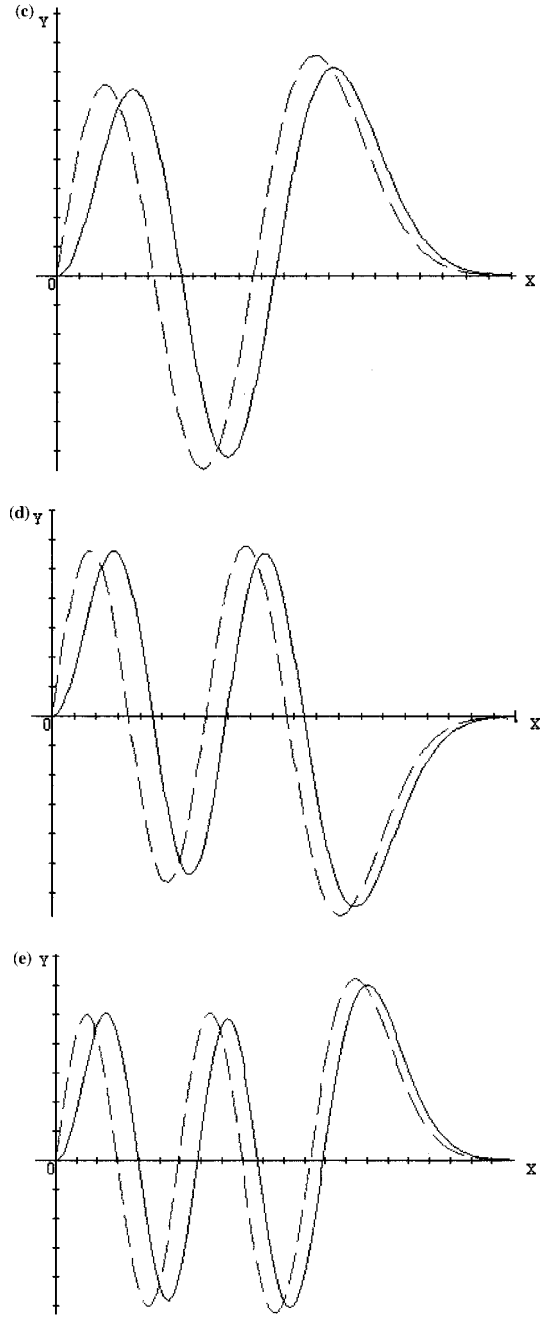

Fig. 1 *Continued*.

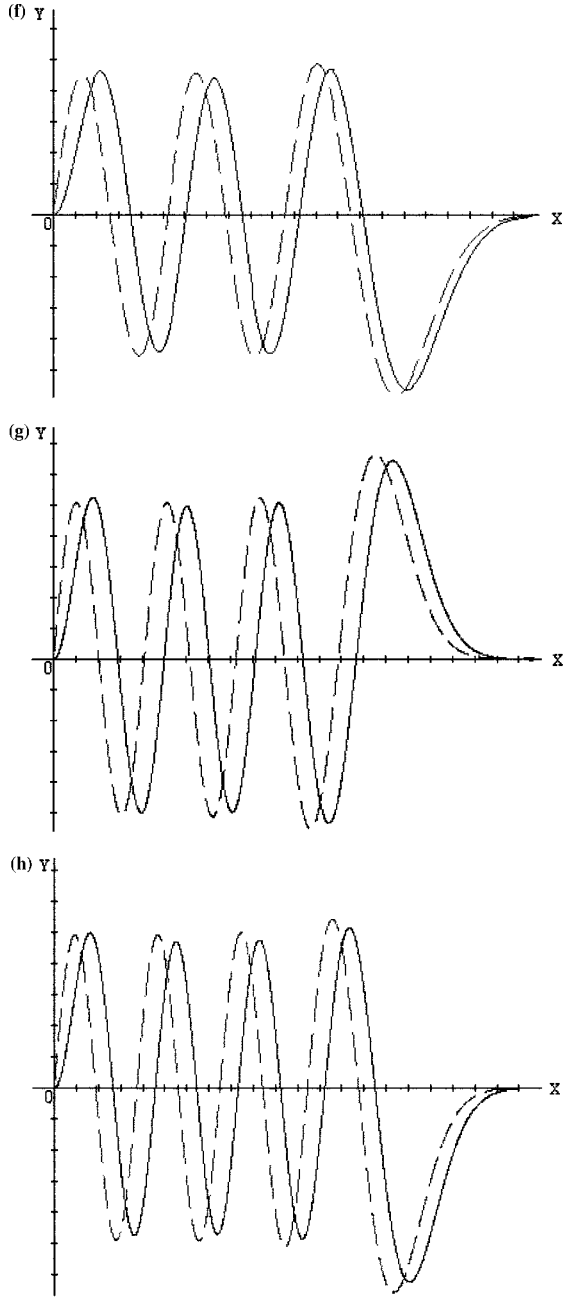

Fig. 1 *Continued.*

Table I. First Low-Order Eigenvalues of the Excited States of a Charged Membrane Under the Influence of an External Magnetic Field, and the Corresponding Energies According to Eqs. (16) and (17), for the Case $j = 1/2$, with $\ell = 0$ and $\ell = 1$

n	$\mathcal{E}_n^{(0)}$	$\mathcal{E}_n^{(1)}$	$E_n^{(0)}$ (MeV)	$E_n^{(1)}$ (MeV)
1	0.2388325	0.2870914	22.3114	26.8196
2	0.4176365	0.4519193	39.0150	42.2176
3	0.5637745	0.5894486	52.6670	55.0654
4	0.6929367	0.7127682	64.7331	66.5858
5	0.8110007	0.8265460	75.7625	77.2147
6	0.9210154	0.9332315	86.0399	87.1811
7	1.0248086	1.0343377	95.7361	96.6263
8	1.1235862	1.1308813	104.9638	105.6453

where $m_e = 0.5110$ MeV is the electron rest mass and

$$\int_0^1 \sqrt{1-u^4}\, du = \frac{\Gamma(1/4)\,\Gamma(3/2)}{4\,\Gamma(7/4)} = 087401918. \tag{21}$$

In this case the first eight excited energy levels are given in Table III, and we see that the mass spectrum predicted by Eq. (20) is also rich, whose values are close to those shown in Table I. By comparing Tables I and III we see that the first excited state ($n = 1$) turns out to be $m \approx 53 m_e$, which is about a quarter of the observed muon rest mass. Such a result has been already derived by Dirac (1962), pointing out to his disappointment that in the membrane model the muon is not the first excited state of the electron. The next step should be to develop an accurate formulation to determine the lifetime of each excited state of the whole spectrum. This is a subject for future investigation.

Table II. Mass Spectrum of the First Eight High-Order Excited Levels of the Dirac Membrane Model, According to the Asymptotic Approximation

m (level number)	Mass (MeV)
1	293.48
2	586.96
3	880.45
4	1173.93
5	1467.41
6	1760.89
7	2054.38
8	2347.86

Table III. Mass Spectrum of the First Eight Excited Levels of the Dirac Membrane Model, According to the Bohr–Sommerfeld Quantization Rule

n (level number)	Mass (MeV)
1	26.7653313
2	42.4873150
3	55.6741326
4	67.4444086
5	78.2623034
6	88.3771765
7	97.942529
8	107.0613252

5. CONCLUDING REMARKS

We have considered in this paper a charged spinning membrane in an external magnetic field within a Dirac formulation, taking into account a spin-rotation coupling term. According to this approach the membrane has quantized radial modes of oscillations that yield a discrete mass–energy spectrum. This result can help us to a deeper understanding of the charged membrane model in a quantum framework. We have also elucidated to a good accuracy the first low-order levels of the energy spectrum for the excited states of the charged membrane, for $j = 1/2$, with $\ell = 0$ and $\ell = 1$, pointing out the nonlinear behavior of this spectrum as a reflection of the essentially nonharmonic nature of the classical motion of the membrane. Moreover, according to Tables I and III, we see that two consecutive energy levels of the membrane become closer with the increasing level number n.

Otherwise, it is reasonable to expect that the energy enclosed by a charged spinning membrane, which is supposed to picture an extended charged lepton, may contribute to its whole action. Furthermore, with a membrane approach as a basis for a relativistic quantum theory of extended massive leptons, we expect that several important questions can be elucidated. We hope the approach discussed here will provide a basis for the development of a more general theory.

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